# Elementary proof of the bound on the speed of quantum evolution

Piotr Kosiński*, Magdalena Zych*
Department of Theoretical Physics II,
Institute of Physics, University of Łódź,
ul. Pomorska 149/153, 90 - 236 Łódź, Poland.

**Abstract**

An elementary proof is given of the bound on "orthogonalization time": $t_0 \geq \frac{\pi\hbar}{2\Delta E}$.

*supported by the Łódź University Grants $N^0$ 690 and 795

In many problems of quantum theory (like, for example, quantum computing [1]$\div$[4] or fidelity between two quantum states [5], [6]) it appears important to estimate speed of quantum evolution.

An interesting measure of evolution speed is provided by the minimum time $t_0$ required for the state to be transformed into an orthogonal (i.e. distinguishable) state. The basic estimate concerning $t_0$ is given by the inequality

$$t_0 \geq \frac{\pi\hbar}{2\Delta E} \tag{1}$$

which has been derived and studied by many authors [7] $\div$ [13]. This bound, in terms of energy dispersion $\Delta E$ of initial state, is very simple and natural (in particular, $\Delta E = 0$ implies $t_0 = \infty$ as it should be since the initial state is then an energy eigenstate). It has been generalized in various directions [14], [15], [5]; also, a beautiful geometric interpretation in terms of Fubini - Study metric was given [16] (see also [17]) and the intelligent states saturating (1) were found [18].

Quite unexpectedly, few years ago Margolus and Levitin [1] derived a new bound of the form

$$t_0 \geq \frac{\pi\hbar}{2(E - E_0)} \tag{2}$$

valid for Hamiltonians bounded from below; here $E_0$ is the lowest energy while $E$ is the expectation value of the Hamiltonian. They were able to show that, for a large class of states, eq. (2) provides a more optimal bound than eq. (1) (on the other hand, for energy eigenstates, except the lowest one, (2) is useless). The intelligent states for the inequality (2) were found in Refs. [19], [20].

While the standard proof of the bound (1) is based on Heisenberg equations of motion and uncertainty principle ( see, however, [12]), Margolus - Levitin derivation of the new bound (2) is surprisingly elementary; moreover, the corresponding intelligent states can be easily found [20].

The question arises whether the bound (1) can be derived along the same lines. The aim of the present note is to provide the positive answer to this question. We shall show that (1) holds provided the Hamiltonian $H$ is selfadjoint and the initial state belongs to its domain. No further restrictions on the properties of $H$ are necessary; in particular, the spectrum may include both discrete and continuous parts and may extend to infinity in both directions.

Let us first sketch a generalization of the elegant approach of Ref.[1]. We assume for simplicity that the spectrum of $H$ is purely discrete; the general case is briefly discussed in the final part of the paper.

Let $\{\mid n\rangle\}$ be the basis consisting of eigenstates of the Hamiltonian $H$,

$$H \mid n\rangle = E_n \mid n\rangle \tag{3}$$

and let

$$| \Psi(0)\rangle = \sum_n c_n \mid n\rangle \tag{4}$$

be some initial state. Then

$$\langle\Psi(0) \mid \Psi(t)\rangle = \sum_n \mid c_n \mid^2 e^{\frac{-iE_n}{\hbar}t} = \left\langle cos(\frac{Ht}{\hbar})\right\rangle_0 - i\left\langle sin(\frac{Ht}{\hbar})\right\rangle_0 \tag{5}$$

here $\langle f(H)\rangle_0 \equiv \sum_n f(E_n) \mid c_n \mid^2$ denotes the average with respect to the initial state. Now, due to $\langle\Psi(0) \mid \Psi(t_0)\rangle = 0$ one obtains

$$\left\langle cos(\frac{Ht_0}{\hbar})\right\rangle_0 = 0, \quad \left\langle sin(\frac{Ht_0}{\hbar})\right\rangle_0 = 0 \tag{6}$$

or

$$\left\langle Acos(\frac{Ht_0}{\hbar} + \alpha)\right\rangle_0 = 0 \tag{7}$$

for arbitrary constants $A, \alpha$.

Consider now an inequality of the form

$$f(x) \geq Acos(x + \alpha) \tag{8}$$

which is assumed to hold for $-\infty < x < \infty$ or $0 \leq x \leq \infty$ if the spectrum of $H$ extends in both directions or is nonnegative, respectively. Then

$$\left\langle f(\frac{Ht}{\hbar})\right\rangle_0 \geq \left\langle Acos(\frac{Ht}{\hbar} + \alpha)\right\rangle_0 \tag{9}$$

provided the left-hand side is well defined ( i.e. average exists). Now, due to eq. (7),

$$\left\langle f(\frac{Ht_0}{\hbar})\right\rangle_0 \geq 0 \tag{10}$$

The above inequality imposes certain restrictions on $t_0$. By a judicious choice of $f(x)$ one can learn something interesting about $t_0$. For example, the bound (2) is obtained taking the optimal inequality (8) in the class of linear functions $f(x)$ (in this case we have to restrict the range of $x$ to positive semiaxis).
Let us now consider (8) in the class of quadratic functions $f(x)$ and $-\infty < x < \infty$. It is an elementary task to check that the optimal inequality reads now

$$(x + \alpha)^2 - \frac{\pi}{4} \geq -\pi cos(x + \alpha) \tag{11}$$

By assumption, $\mid \Psi(0)\rangle$ belongs to the domain of $H$ and both $\langle H\rangle_0$ and $\langle H^2\rangle_0$ are well defined [21]. Eq. (10) takes now the form

$$\frac{\langle H^2\rangle_0}{\hbar^2}t_0^2 + \frac{2\alpha\langle H\rangle_0}{\hbar}t_0 + (\alpha^2 - \frac{\pi^2}{4}) \geq 0 \tag{12}$$

which implies that $t_0$ lies outside the open interval

$$\Delta_\alpha \equiv \left( \frac{-2\alpha\langle H\rangle_0 - \sqrt{\pi^2\langle H^2\rangle_0 - 4\alpha^2\Delta E_0^2}}{\frac{2\langle H^2\rangle_0}{\hbar}} , \frac{-2\alpha\langle H\rangle_0 + \sqrt{\pi^2\langle H^2\rangle_0 - 4\alpha^2\Delta E_0^2}}{\frac{2\langle H^2\rangle_0}{\hbar}} \right) \tag{13}$$

where $\Delta E_0^2 \equiv \langle H^2\rangle_0 - \langle H\rangle_0^2$. It follows from eq.(13) that $\Delta_\alpha$ is nonempty provided $\alpha$ belongs to the open interval

$$\Omega \equiv \left( \frac{-\pi\sqrt{\langle H^2\rangle_0}}{2\Delta E_0} , \frac{\pi\sqrt{\langle H^2\rangle}}{2\Delta E_0} \right) \tag{14}$$

So, finally, we obtain

$$t_0 \notin \bigcup_{\alpha\in\Omega} \Delta_\alpha = \left( \frac{-\pi\hbar}{2\Delta E_0} , \frac{\pi\hbar}{2\Delta E_0} \right) \tag{15}$$

which implies (1).

In order to find intelligent states for the bound (1) we define

$$\gamma_\alpha(x) \equiv (x+\alpha)^2 - \frac{\pi^2}{4} + \pi cos(x+\alpha) \tag{16}$$

Then

$$\gamma_\alpha(x) \geq 0 \tag{17}$$

and $\gamma_\alpha(x) = 0$ if and only if $x = -\alpha \pm \frac{\pi}{2}$.
Assuming $t_0 = \frac{\pi\hbar}{2\Delta E_0}$ we find from (12) and (16)

$$\left\langle \gamma_\alpha(\frac{Ht_0}{\hbar}) \right\rangle_0 = 0 \quad for \quad \alpha = \frac{-\pi\langle H\rangle_0}{2\Delta E_0} \tag{18}$$

Now, due to (17), eq.(18) implies $c_n \neq 0$ only if $\frac{E_n t_0}{\hbar} = \frac{\pi\langle H\rangle_0}{2\Delta E_0} \pm \frac{\pi}{2}$. Therefore, $c_n \neq 0$ for at most two levels and $E_{n_1} = \langle H\rangle_0 + \Delta E_0,\ \ E_{n_2} = \langle H\rangle_0 - \Delta E_0$ which holds provided $\mid c_{n_1}\mid^2 = \mid c_{n_2}\mid^2 = \frac{1}{2}$. Therefore, the intelligent states are of the form [18]

$$\mid\chi\rangle = c_1 \mid n_1\rangle + c_2 \mid n_2\rangle, \quad \mid c_1\mid^2 = \mid c_2\mid^2 = \frac{1}{2} \tag{19}$$

Finally, let us briefly discussed the general case when no assumption concerning the spectrum of $H$ is made. Spectral theorem [21] allows us to write

$$\langle\Psi(0)\mid\Psi(t)\rangle = \langle\Psi(0)\mid e^{\frac{-it}{\hbar}H}\mid\Psi(0)\rangle = \int e^{\frac{-iEt}{\hbar}} d\langle\Psi(0)\mid P_E\mid\Psi(0)\rangle \tag{20}$$

where $P_E$ is a spectral measure for energy. By assumption $\mid \Psi(0)\rangle$ belongs to the domain of $H$ which implies [21]

$$\int E^2 d\langle\Psi(0) \mid P_E \mid \Psi(0)\rangle < \infty \tag{21}$$

Therefore, $\gamma_\alpha(\frac{Et}{\hbar})$ is integrable and

$$\int \gamma_\alpha(\frac{Et}{\hbar}) d\langle\Psi(0) \mid P_E \mid \Psi(0)\rangle \geq 0 \tag{22}$$

which again leads to the estimate (1).

# References


[1] N. Margolus, L.B. Levitin, Physica **D120** (1998), 188

[2] A. Miyake, M. Wadati, Phys. Rev. **A64** (2001), 042317

[3] S. Lloyd, Phys. Rev. Lett. **88** (2002), 237901

[4] S. Lloyd, quant - ph /9908043

[5] V. Giovannetti, S. Lloyd, L. Maccone, Phys. Rev. **A67** (2003), 052109

[6] V. Giovannetti, S. Lloyd, L. Maccone, Proc. of the SPIE, vol. 5111 (2003), 1

[7] L. Mandelstam, I. Tamm, Journ. Phys. (USSR) **9** (1945), 249

[8] G.N. Fleming, Nuovo Cim. **A16** (1973), 232

[9] K. Bhattacharyya, Journ. Phys. **A16** (1983), 2991

[10] D. Home, M.A.B. Whitaker, Journ. Phys. **A19** (1986), 1847

[11] A. Peres, Quantum Theory: Concepts and Methods, Kluwer, Hingham 1985

[12] L. Vaidman, Am. Journ. Phys. **60** (1992), 182

[13] L. Vaidman, O. Belkind, quant - ph /9707060

[14] P. Pfeifer, Phys. Rev. Lett. **70** (1993), 3365

[15] J. Uffink, Am. Jour. Phys. **61** (1993), 935

[16] J. Anandan, Y. Aharonov, Phys. Rev. Lett. **65** (1990), 1697

[17] D.C. Brody, Journ. Phys. **A36** (2003), 5587

[18] N. Horesh, A. Mann, Journ. Phys. **A31** (1998), L609

[19] J. Söderholm, G. Björk, T. Tsegaye, A. Trifonov, Phys. Rev. **A59** (1999), 1788

[20] K. Andrzejewski, unpublished (2005)

[21] M. Reed, B. Simon, "Methods of Modern Mathematical Physics", vol.1, Academic Press New York, Londyn,1972